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SMALL DEFORMATIONS OF A PLASTIC-RIGID BODY

AT THE YIELD POINT

by

E. T. Onat

GRADUATE DIVISION OF APPLIED MATHEMATICS

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PREFACE

The author of this paper, Dr. E. T. Onat, is a F. B.
Jewett Fellow in Applied Mathematics at Brown University. Even
though the paper has not been written under Contract N7onr-
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E. H. Lee, Chairman
Graduate Division of
Applied Mathematics

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RIGID BODY AT THE YIELD POINT
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1. Introduction. Consider a body of plastic-rigid material
which is loaded by monotonically increased surface tractions.
Let us assume that the history of loading is completely specified.
At the earlier stages of loading the body will generally remain
rigid. If the loads are increased further some portions of the
body may become locally plastic. However, plastic deformation
of these portions may be prevented by adjacent rigid portions
until the moment when the body actually begins to deform. Follow-
ing Hill [1]#, we refer to this critical moment as the yield
point of the plastic-rigid body. If the material is perfectly
plastic (i.e., non-strain-hardening) and if the accompanying
change in geometry is disregarded, plastic flow is found to con-
tinue under constant loads [2]. However, when the change in
geometry and the effects of strain-hardening are taken into ac-
count, continuing deformation under constant external forces is
possible only in exceptional cases [1]. Thus, as a rule, quasi-
static flow requires either increasing or decreasing external
forces. Under the assumed monotonically increasing loading, the
first case represents a stable process, whereas the second case
leads to sudden collapse.

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*   F. B. Jewett Fellow at Brown University, Providence, R. I.

Numbers in square brackets refer to the Bibliography at the
    end of the paper.

The distribution of stress and the corresponding velocity field at the yield-point can be found in some cases (e.g., by the method of characteristics in plane strain problems) and in other cases approximations from above and below to the yield-point load can be obtained [1,2]. However, in many cases the present first order theory does not furnish a clear criterion for collapse. Therefore it seems worthwhile to investigate the second order effects in the deformation of a plastic-rigid body at the yield point.

In the following the investigation is limited to problems where the entire body is in the plastic state at the yield point and where the incipient stress and velocity fields are known. A perturbation scheme is developed to obtain equations governing Lagrangian stress and velocity increments of the subsequent quasi-static motion. Some examples are worked out (compression of block between rough plates and the thick walled tube under internal pressure). In the first case the results obtained with the theory are found to agree with the experimental results. In the second example the minimum rate of strain-hardening necessary to prevent collapse is found.

2. Notations and Stress-strain Relations Employed in the Analysis.
Let x_i denote the rectangular Cartesian coordinates of a generic particle at the time t. If the velocity of this particle is denoted by v_i, the velocity strain tensor is given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (1)$$

where Latin indices take the values 1, 2, 3.

The stress and deviatoric stress tensors are denoted respectively by σ_{ij} and s_{ij}:

$$s_{ij} = \sigma_{ij} - (\tfrac{1}{3}\,\sigma_{kk})\delta_{ij}, \qquad (2)$$

where summation convention is used and δ_{ij} represents the Kronecker delta.

If the second invariant of the deviatoric stress tensor is denoted by J_2,

$$J_2 = \tfrac{1}{2}\,s_{ij}s_{ij}. \qquad (3)$$

The particular stress-strain relations considered in this paper are

$$\left.\begin{aligned}
\varepsilon_{ij} &= F(J_2)s_{ij}\frac{DJ_2}{Dt} \quad \text{when} \quad J_2 \geq k^2 \text{ and } \frac{DJ_2}{Dt} \geq 0 \\[2mm]
\varepsilon_{ij} &= 0 \qquad\qquad\quad \text{when} \quad J_2 < k^2 \text{ or } \frac{DJ_2}{Dt} < 0
\end{aligned}\right\} \quad (4)$$

where DJ_2/Dt is the material derivative of J_2.

The limiting case where

$$F(J_2) \longrightarrow \infty \quad \frac{DJ_2}{Dt} \longrightarrow 0$$

defines an ideally plastic material:

$$J_2 = k^2,$$

$$\left.\begin{aligned}
\varepsilon_{ij} &= \lambda s_{ij} \quad \text{when} \quad \lambda \geq 0 \\[2mm]
\varepsilon_{ij} &= 0 \qquad \text{when} \quad \lambda < 0.
\end{aligned}\right\} \quad (5)$$

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$$\left.\begin{aligned}
\varepsilon_{ij} &= \lambda s_{ij} \quad \text{when} \quad \lambda \geq 0\\
\varepsilon_{ij} &= 0 \quad\quad \text{when} \quad \lambda < 0.
\end{aligned}\right\} \quad (5)$$

3. Formulation of the Problem. Consider a plastic-rigid body
at the yield point. Suppose that the boundary conditions are
such that every element of the material is in the plastic state.
Moreover assume that a velocity field v_i^0 is associated with the
stress-state σ_{ij}^0 where v_i^0 and σ_{ij}^0 are the Cartesian components
of the velocity and stress tensors. v_i^0 and σ_{ij}^0 satisfy the bound-
ary conditions imposed on the displacements and the stresses in
addition to the following equations:

$$\sigma_{ij,j}^0 = 0, \qquad \text{(Equilibrium)} \qquad (6)$$

$$s_{ij}^0 s_{ij}^0 = 2k^2, \qquad \text{(Yield condition)} \qquad (7)$$

$$\tfrac{1}{2}(v_{i,j}^0 + v_{j,i}^0) = \epsilon_{ij}^0 = \lambda s_{ij}^0 \qquad \begin{array}{l}\text{(Stress strain rela-} \\ \text{tion)}\end{array} \qquad (8)$$

where
$$\lambda > 0.$$

We note that this solution is valid for an ideally plas-
tic material and also for a newly annealed rigid-strain-hardening
material which has the same yield stress, k, in pure shear.

Now assume that the surface tractions are varied in such
a manner that the further distortion following the reaching of
the yield point takes place in a quasi-static manner.

We now consider the early stages of the distortion such
that the state of velocity-strain and stress differs only slightly
from the state which is given by Eqs. (6), (7), and (8).

Let x_i be the Cartesian coordinates of the material point
a_p at time t:

$$x_i = x_i[a_p, \delta(t)] \qquad (9)$$

where $\delta(t)$ is a monotonously increasing function of time and

$$\delta(0) = 0, \quad x_i[a_p, 0] = a_i.$$

If the velocity and acceleration at time zero are continuous functions of the space variables, the validity of the following expansion may be assumed:

$$x_i = a_i + v_i^o(a_p)\delta + \gamma_1(a_p)\frac{\delta^2}{2} + \dots \tag{10}$$

where to obtain a quasi-static flow, $d\delta/dt$; $d^2\delta/dt^2$ are supposed to be as small as desired. Since the stress-strain relations envisaged in this theory are homogeneous in time, the actual form of these functions is of no importance in the problem considered.

The velocity of the material point a_p at time δ referred to the fixed rectangular Cartesian coordinates is then given by

$$v_i = (v_i^o + \gamma_1\delta + \dots)\frac{d\delta}{dt}. \tag{11}$$

The velocity strain tensor defined in the Eulerian manner is

$$\varepsilon_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) = \frac{1}{2}(\frac{\partial v_i}{\partial a_p}\frac{\partial a_p}{\partial x_j} + \frac{\partial v_j}{\partial a_p}\frac{\partial a_p}{\partial x_i}) \dots. \tag{12}$$

From (10), $\partial a_p/\partial x_j$ can be evaluated (see Appendix) in terms of v_i^o:

$$\frac{\partial a_p}{\partial x_j} = \delta_{pj} - \delta v_{p,j}^o + \dots \tag{13}$$

where a comma denotes partial differentiation with respect to a_j.

From the last three equations, we obtain

$$\varepsilon_{ij} = [\varepsilon_{ij} + \delta\varepsilon_{ij}' + \dots]\frac{d\delta}{dt} \tag{14}$$

where $\varepsilon_{ij}^o = \frac{1}{2}(v_{i,j}^o + v_{j,i}^o)$ and

$$\epsilon'_{1j} = \tfrac{1}{2}(\gamma_{1,j} + \gamma_{j,1}) - \tfrac{1}{2}(v^0_{1,p}v^0_{p,j} + v^0_{j,p}v^0_{p,1}). \quad (15)$$

Let $\sigma_{1j}(a_p,\delta)$ be the components of the stress tensor at the point $x_1 = x_1(a_p,\delta)$, referred to the fixed rectangular Cartesian coordinates. For small deviations from the yield point state we again assume that

$$\sigma_{1j} = \sigma^0_{1j}(a_p) + \delta\sigma'_{1j}(a_p) + \dots . \quad (16)$$

We now examine conditions and equations which must be satisfied during the quasi-static distortion of the body after the yield point.

a. Equations of Equilibrium: Since a quasi-static motion of the body is considered, the Equations of Equilibrium must be satisfied during the subsequent flow:

$$\frac{\partial\sigma_{1j}}{\partial x_j} = \frac{\partial\sigma_{1j}}{\partial a_p} \cdot \frac{\partial a_p}{\partial x_j} = 0, \quad (17)$$

where body forces are neglected.

Using (13) and (16) this becomes

$$\frac{\partial\sigma^0_{1j}}{\partial a_j} + \delta\left[\frac{\partial\sigma'_{1j}}{\partial a_j} - v^0_{p,j}\frac{\partial\sigma^0_{1j}}{\partial a_p}\right] = 0 \quad (18)$$

where only the terms up to the order δ are retained. The first term of this equation vanishes according to (6), hence the Equilibrium Equations satisfied by the Lagrangian stress increments are

$$\frac{\partial\sigma'_{1j}}{\partial a_j} - v^0_{p,j}\frac{\partial\sigma^0_{1j}}{\partial a_p} = 0. \quad (19)$$

b. <u>Stress-Strain Relations.</u> An investigation of the stress-
strain relations at time zero and time δ provides six additional
equations for Lagrangian Stress- and Strain increments.

At time $\delta = 0$,

$$\varepsilon_{ij}^{O} = F(J_2) s_{ij}^{O} \frac{dJ_2(a_p,\delta)}{d\delta} \tag{20}$$

where

$$2J_2(a_p,\delta) = s_{ij}^{O} s_{ij}^{O} + 2\delta s_{ij}^{O} s_{ij}^{'} + \cdots$$

and $s_{ij}^{'}$ denote increments of the deviatoric stress tensor. There-
fore if the terms up to the order of unity are retained,

$$\frac{dJ_2}{d\delta} = s_{pq}^{O} s_{pq}^{'}$$

and (20) gives

$$s_{pq}^{O} s_{pq}^{'} = \frac{\varepsilon_{ij}^{O}}{s_{ij}^{O} F(J_2^{O})} \tag{21}$$

where the right hand side is not summed and any combination of
(i,j) may be chosen provided that $s_{ij}^{O} \neq 0$.

If the material is ideally plastic (21) becomes

$$s_{pq}^{O} s_{pq}^{'} = 0. \tag{22}$$

At time δ, in addition to the equation of incompressibility, the
stress-strain relations provide four independent equations of
the following type:

$$\frac{\varepsilon_{ij}}{s_{ij}} = \frac{\varepsilon_{pq}}{s_{pq}} , \tag{23}$$

where again the summation convention does not apply. Using (8),
(15) and the definition of $s_{ij}^{'}$, and retaining terms up to the
order of δ one obtains the following equations

$$\epsilon_{ij}' s_{pq}^{o} + \epsilon_{ij}^{o} s_{pq}' - \epsilon_{pq}^{o} s_{ij}' - \epsilon_{pq}' s_{ij}^{o} = 0, \qquad (24)$$

where (i,j,p,q) are to be chosen in order to give four independent equations.

On the other hand the equation of incompressibility, which is obtained from the contraction of the tensorial equation (4), gives

$$\epsilon_{ii} = 0,$$

and using (14) and (15), and observing that $\epsilon_{ii}^{o} = 0$, we get

$$\Upsilon_{i,i} - v_{i,p}^{o} v_{p,i}^{o} = 0. \qquad (25)$$

We note that Eqs. (19), (21) (or 22), (24) and (25) provide nine <u>linear</u> equations for the nine unknowns σ_{ij}' and Υ_i. In the case of plane strain, Eqs. (19) and (21) or (22) provide three independent equations for three unknown stress-increments. Therefore if the boundary conditions are of the stress type the problem is statically determined.

c. <u>Boundary Conditions</u>. We first examine conditions which must be satisfied at a boundary surface, S_F, of the body at which surface forces are prescribed. Denote by Q_i the components of the surface tractions in the fixed rectangular Cartesian coordinate system:

$$Q_i(a_p, \delta) = Q_i^{o}(a_p) + \delta Q_i'(a_p) + \dots . \qquad (26)$$

Suppose that S_F is defined by the following equation

$$F(a_1, a_2, a_3) = 0 \qquad (27)$$

where as before

$$x_1 = x_1(a_p, \delta)$$

or

$$a_p = a_p(x_1, \delta)$$

and it is assumed that points originally on the boundary surface remain on a surface during the deformation. At time δ, the unit normal n_p of the surface S_F has direction cosines proportional to $\partial F/\partial x_p$. Using the series expansions and (15) one obtains:

$$n_p = n_p^0 + \delta n_p' + \ldots = k \frac{\partial F}{\partial x_p} = k \frac{\partial F}{\partial a_r}(\delta_{rp} - \delta v_{r,p}^0),$$

or

$$n_p^0 + \delta n_p' = k(\frac{\partial F}{\partial a_p} - \delta F_{,r} v_{r,p}^0). \tag{28}$$

The magnitude of k must be such that

$$n_p n_p = 1 \qquad \text{(unit normal).} \tag{29}$$

From (13) and (14)

$$k = \pm \frac{1}{\sqrt{F_{,r}F_{,r}}} \left\{ 1 + \delta \frac{F_{,p}F_{,r}v_{r,p}^0}{F_{,j}F_{,j}} \right\} \tag{30}$$

where the sign will be properly chosen to obtain the outward normal of the surface.

On the other hand, the surface tractions are related to the stresses by

$$Q_1 = n_p \sigma_{p1}$$

or

$$Q_1^0 + \delta Q_1' + \ldots = (\sigma_{p1}^0 + \delta \sigma_{p1}' + \ldots)(n_p^0 + \delta n_p' + \ldots),$$

$$Q_1' = \sigma_{p1}' n_p^0 + \sigma_{p1}^0 n_p'.$$

From (28) and (30);

$$n_p^o = \pm \frac{F_{,p}}{\sqrt{F_{,j}F_{,j}}} \, ,$$

$$n_p' = \pm \frac{1}{\sqrt{F_{,j}F_{;j}}} \left[F_{,p} \frac{F_{,q}F_{,r}v_{r,q}^o}{F_{,k}F_{,k}} - F_{,r}v_{r,p}^o \right]$$

and therefore the increments of the surface tractions are related to the increments of the stresses by the following formula:

$$Q_i' = \sigma_{pi}' \frac{F_{,p}}{\sqrt{F_{,j}F_{,j}}} \mp \frac{\sigma_{pi}^o}{\sqrt{F_{,j}F_{,j}}} \left[F_{,p} \frac{(F_{,q}F_{,r}v_{r,q}^o)}{F_{,j}F_{,j}} - F_{,r}^o v_{r,p}^o \right]. \quad (31)$$

As to the displacement boundary conditions, we infer that on S_v, Y_i is given as functions of the original coordinates.

The theory has now been taken to a point from which applications may be made to specific problems.

Example 1.   Compression of Block Between Rough Plates:

Consider a block of strain-hardening (or ideally plastic)-rigid material compressed between rough plates (Fig. 1).

When the block is very wide compared with its height, the slip line field corresponding to the incipient motion is independent of x, and the incipient stress and velocity fields are given by Prandtl [3] and Nadai [4]:

$$\left.\begin{array}{l} \dfrac{\sigma_x^o}{k} = -c - \dfrac{mx}{h} + 2\sqrt{1 - m^2 \dfrac{y^2}{h^2}} \\[3mm] \dfrac{\sigma_y^o}{k} = -e - \dfrac{mx}{h} \\[3mm] \dfrac{\tau_{xy}^o}{k} = \dfrac{my}{h} \end{array}\right\} \qquad (32)$$

and

$$\left. \begin{array}{l} \dfrac{u}{U} = C + \dfrac{x}{h} - \dfrac{2}{m}\sqrt{1 - m^2 \dfrac{y^2}{h^2}} \\[3mm] \dfrac{v}{V} = - \dfrac{y}{h}\,, \end{array} \right\} \tag{33}$$

where k is the yield stress in pure shear, m is a measure of roughness $[0 \leq m \leq 1;\ m = 1$ for perfectly rough plates] and

$$\left. \begin{array}{l} c = \dfrac{\sin^{-1}m}{m} + \sqrt{1 - m^2}\,, \\[3mm] C = - \dfrac{L}{h} + \dfrac{\sin^{-1}m}{m^2} + \dfrac{1}{m}\sqrt{1 - m^2}\,. \end{array} \right\} \tag{34}$$

These equations state that when the normal pressure, p, applied by the plates reaches the value given by

$$\frac{p}{k} = c + \frac{mx}{h} + 2\sqrt{1 - m^2}$$

the plastic rigid block begins to deform according to the velocity field (33). As stated before we are interested in the subsequent neighboring deformation of the body and in the rate of increase of the external forces to achieve this deformation.

If (x,y) are regarded as Lagrangian coordinates and if the current coordinates are denoted by $(\bar{x},\bar{y})$,

$$\bar{x} = x + \delta\frac{u}{V} + \cdots$$

$$\bar{y} = y + \delta\frac{v}{U} + \cdots$$

where δ denotes the compression.

Equations (19) and (21) provide three independent relations to determine the Lagrangian stress increments σ_x', σ_y', τ_{xy}'.

From (19), (32) and (33)

$$\frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau'_{yx}}{\partial y} + \frac{2km}{h^2} = 0$$

$$\frac{\partial \sigma'_y}{\partial y} + \frac{\partial \tau'_{yx}}{\partial x} = \frac{k}{h} f,y \qquad \qquad (35)$$

where

$$f(y) = 2 \sqrt{1 - m^2 \frac{y^2}{h^2}} \;.$$

From (21)

$$\sigma'_x - \sigma'_y + 4\tau'_{xy} \frac{y}{hf} = - \frac{f,y}{m^2 k^2 F} \; \frac{h}{y} \;. \qquad (36)$$

If it is assumed that $(\sigma'_x - \sigma'_y)$ and τ'_{xy} are functions of y only, the above equations, with the boundary conditions $\tau_{xy} = 0$ when $y = 0$, and $\tau_{xy} = mk$ when $y = h$, give

$$\tau'_{xy} = 0$$

$$\sigma'_y = \frac{k}{h} f - 2k \frac{m}{h^2} x + b \qquad \qquad (37)$$

$$\sigma'_x = \sigma'_y - \frac{f,y}{m^2 k^2 F} \; \frac{h}{y}$$

where b is an undetermined constant. To determine b let us consider the surface AB which after a small deformation becomes a plane, A'B', parallel to the y-axis, Fig. 1.

Transformation from AB to A'B' is given by

$$\frac{x}{h} + \frac{\delta}{h} (C + \frac{x}{h} - \frac{f}{m}) = \frac{\overline{x}}{h} = \text{constant}$$

$$\frac{y}{h} (1 - \frac{\delta}{h}) \qquad \qquad = \frac{\overline{y}}{h} \;. \qquad \qquad (38)$$

Since the left hand edge of the block is free from stress, the overall equilibrium condition demands that

$$\int_0^{h(1-\delta/h)} \sigma_x d\overline{y} = - mk\overline{x}. \qquad (39)$$

Using (32), (37), (38) the last equation gives

$$mkC + bh - \frac{h}{m^2k^2F} \int_0^h \frac{f,y}{y}\, dy = 0,$$

hence

$$mkC + bh = - \frac{2}{mk^2F}\ \text{arc sin } m. \tag{40}$$

The normal stress in y-direction is then obtained from (37) and (40):

$$\sigma_y = \sigma_y^0 + \delta\sigma_y' = (f-c)k - mk\, \frac{\bar{x}}{h}\,[1 + \frac{\delta}{h}] - \frac{2\delta}{h}\ \frac{\text{arc sin } m}{mk^2F}. \tag{41}$$

The average normal pressure required to continue the deformation is therefore

$$p = - (2\ \sqrt{1 - m^2} - c)k + \frac{mkL}{2h}\,[1 + \frac{\delta}{h}] + \frac{2\delta}{h}\ \frac{\text{arc sin } m}{mk^2F}. \tag{42}$$

In order to compare this result with experiment [5], let us consider the case m = 1:

$$\frac{p}{k} = \frac{\pi}{2} + \frac{L}{2H} + \frac{\delta}{h}\ [\frac{\pi}{k^3F} + \frac{L}{2H}]. \tag{43}$$

The dimensionless quantity Fk^3 can be obtained from the stress-strain curve in compression:

$$Fk^3 = \frac{\sqrt{3}}{4}\ \frac{\sigma}{(\frac{d\sigma}{d\varepsilon})}$$

where σ is the yield stress in compression and $d\sigma/d\varepsilon$ is the slope of the stress-strain curve.

The stress-strain curve of the material used in the experiments is shown in the Fig. 2. The material begins to deform plastically at 50 Bars and at this point

$$\frac{\pi}{Fk^3} = \frac{4\pi}{\sqrt{3}}\ \frac{d\sigma/d\varepsilon}{\sigma} \sim 1125.$$

Hence according to (43),[the slope of the load-deformation curve] at the beginning of the deformation is strongly dependent on the rate of strain-hardening since

$$\frac{\pi}{Fk^3} \gg \frac{L}{2H} \ .$$

The results obtained from the experiments agree with this observation (Fig. 3). Moreover the straight line (I) defined by the equation (43) follows closely the experimental results.

In order to have an idea on the shape of load deformation curve, one can approximate the stress-strain relation with an ideally plastic material $(F \rightarrow \infty)$ as shown in Fig. 2. The straight lines (II) defined by (43) are approximate load-deformation curves corresponding to this idealized material (Fig. 3).

Example 2: Cylinder Under Internal Pressure:

A tube made of plastic-rigid material is subjected to internal pressure. The tube is supposed to be long enough so that the axial deformations of the tube can be neglected. If the yield stress of the material is denoted by k, incipient stress and velocity fields are given by (Fig. 4)

$$\sigma_r^0 = 2k \log \frac{r}{b}$$

$$\sigma_\theta^0 = 2k (1 + \log \frac{r}{b}) \tag{44}$$

$$\tau_{r\theta}^0 = 0$$

and

$$\overset{0}{v}(r) = \frac{a}{r} \ .$$

The material points situated on the circle of the radius r would move after time δ to another circle with the radius $\bar{r}$

$$\bar{r} = r + \frac{a}{r} \delta + \dots \ .$$

The physical components of the stress in these elements would be given by

$$\sigma_r = \sigma_r^0 + \delta\sigma_r' + \dots$$

$$\sigma_\Theta = \sigma_\Theta^0 + \delta\sigma_\Theta' + \dots$$

$$\tau_{r\Theta} = \tau_{r\Theta}^0 = 0.$$

The Equilibrium Equation satisfied by the Lagrangian Stress-increments is found either from the tensorial Equation* (21); or from a straight forward derivation:

$$\frac{\partial\sigma_r'}{\partial r} + \frac{\sigma_r' - \sigma_\Theta'}{r} + 4k\frac{a}{r^3} = 0. \tag{45}$$

On the other hand, Equation (16) gives

$$\sigma_\Theta' - \sigma_r' = \frac{a}{r^2 k^2 F} \ . \tag{46}$$

Eliminating σ_Θ' from these Equations

$$\frac{\partial\sigma_r'}{\partial r} - \frac{a}{k^2 F r^3} + \frac{4ka}{r^3} = 0$$

$$\frac{\partial\sigma_r'}{\partial r} + \frac{ak}{r^3} \left[4 - \frac{1}{k^3 F} \right] = 0$$

$$\sigma_r' - \frac{ak}{2r^2} \left[4 - \frac{1}{k^3 F} \right] + c = 0.$$

Using the boundary condition

$$\sigma_r' = 0 \quad \text{at} \quad r = b$$

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* This equation must first be rewritten for curvilinear coordinates.

we get

$$\sigma_r' = \frac{ak}{2}\left[4 - \frac{1}{k^3 F}\right]\left[\frac{1}{r^2} - \frac{1}{b^2}\right]$$

and

$$\sigma_r(r,\delta) = 2k \log \frac{r}{b} + \frac{ak}{2}\left[4 - \frac{1}{k^3 F}\right]\left[\frac{1}{r^2} - \frac{1}{b^2}\right]\delta. \qquad (47)$$

It is easily seen from this equation that if

$$F > \frac{1}{4k^3} \qquad\qquad (48)$$

$$|\sigma_r(a,\delta)| \leq |\sigma_r(a,0)|,$$

hence a quasi-static motion can only be maintained with decreasing pressure. [In the case where $F \to \infty$ (ideally plastic material), (47) checks with a previous analysis [6]]. However if

$$F < \frac{1}{4k^3}$$

deformation can be continued with increasing pressures. In terms of a simple stress-strain relation in compression the inequality (48) can be written

$$\frac{d(\sigma/\sigma_0)}{d\epsilon} < \sqrt{3} \qquad \text{collapse}$$

$$\frac{d(\sigma/\sigma_0)}{d\epsilon} > \sqrt{3} \qquad \text{no collapse.}$$

In Fig. (3), the limiting curve separating the two regions is shown.

Appendix: It is well known from Tensor Analysis [7] that

$$\frac{\partial a^p}{\partial x^1} = g^{pm} \frac{\partial x^1}{\partial a^m} \tag{49}$$

where g^{pm} are the contravariant components of the metric tensor of the curvilinear system defined by a^p in the deformed position. However g^{pm} are given [8] to the desired order of approximation by

$$g^{pm} = \delta_{pm} - \delta(v^o_{p,m} + v^o_{m,p})$$

and since

$$\frac{\partial x^1}{\partial a^m} = \delta_{1m} + v^o_{1,m}\delta + \cdots \;,$$

(49) gives

$$\frac{\partial a^p}{\partial x^1} = \delta_{p1} - \delta v_{p,1} + \cdots \;.$$

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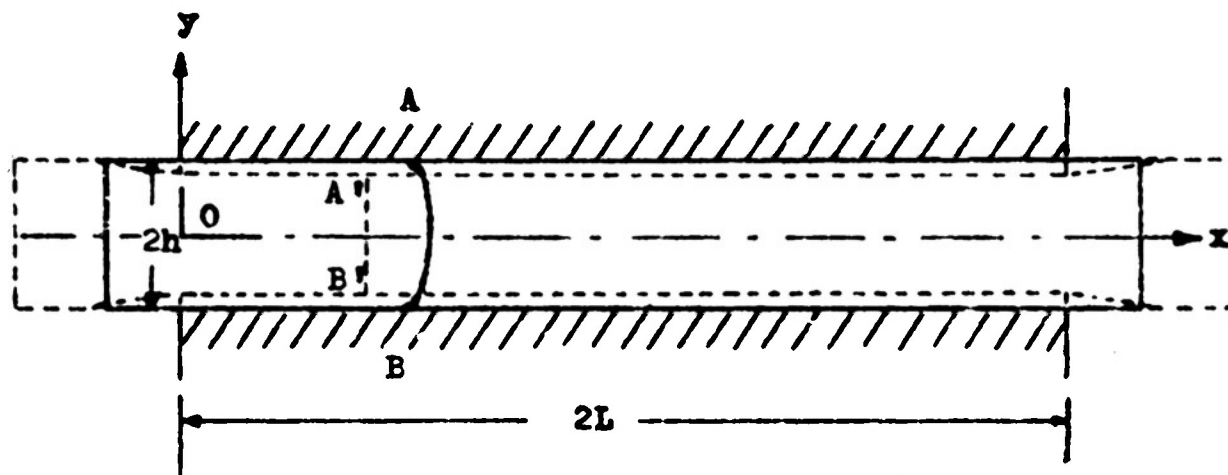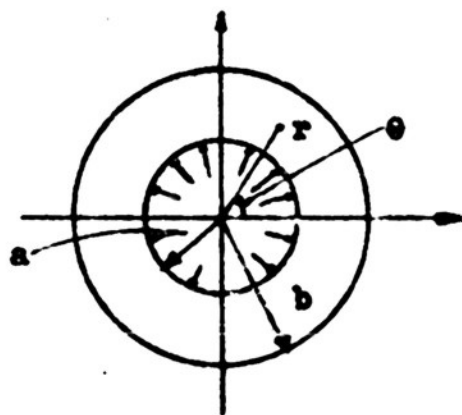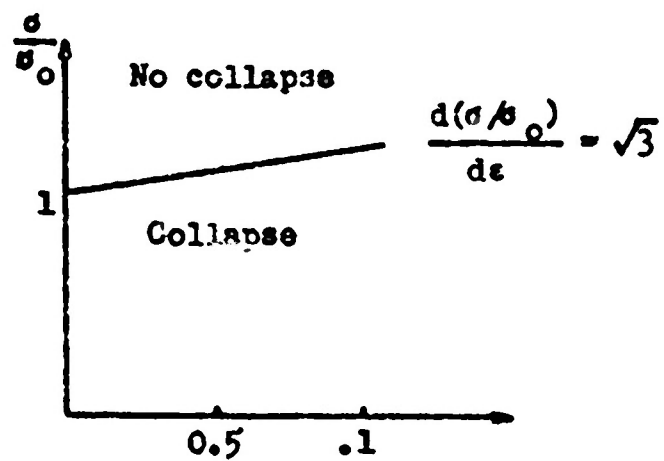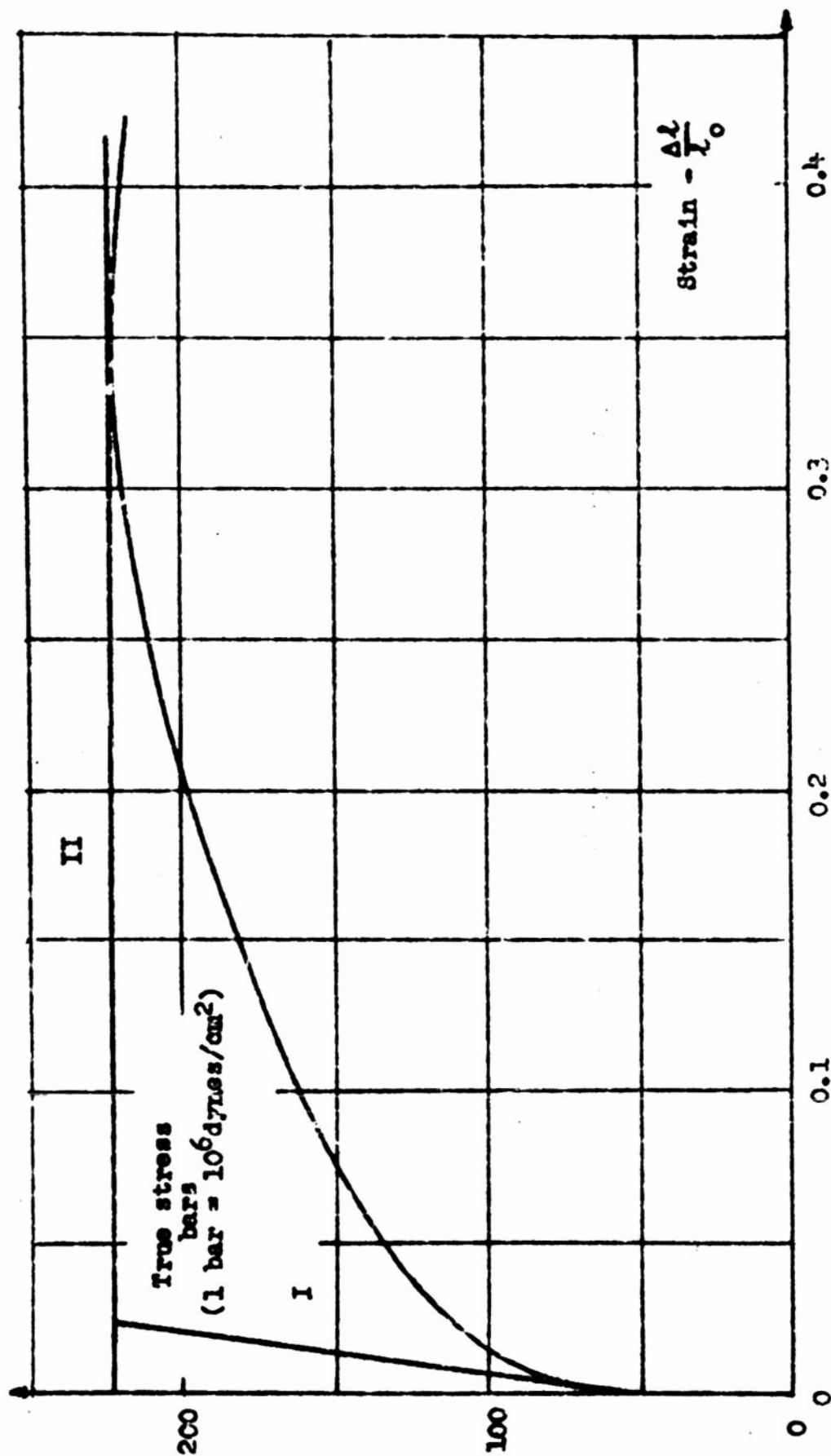   (1952).

Fig. 1

Fig. 4

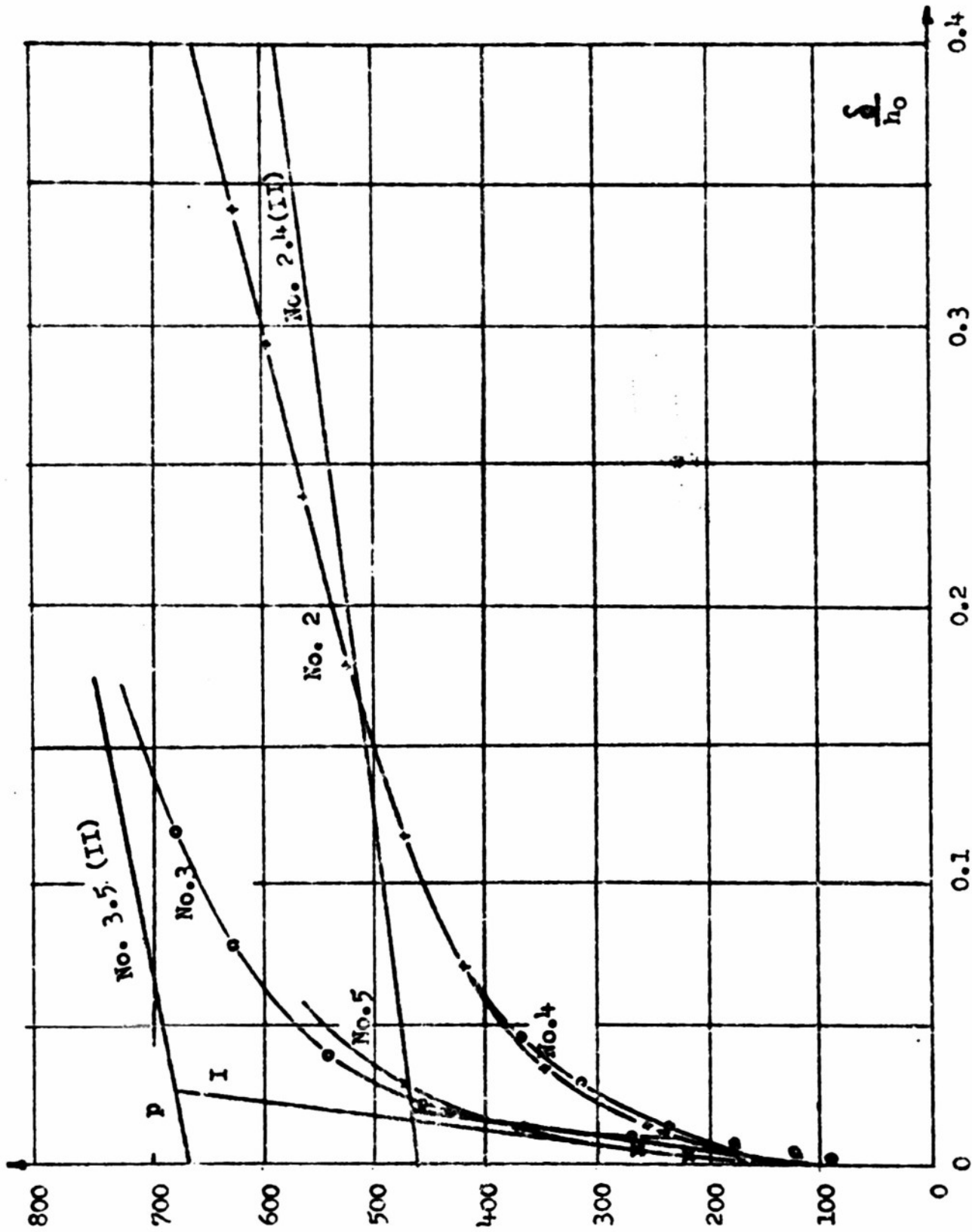Fig. 5

Fig. 2. Mean stress-strain curve in compression

Fig. 3. Mean pressure on plates as a function of amount of compression.